

# Fluid/Gravity duality with Petrov-like boundary condition in a spacetime with a cosmological constant

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## Abstract

Recently it has been shown that imposing Petrov-like condition on the boundary may reduce the Einstein equation to the Navier-Stokes equation in the non-relativistic and near-horizon limit. In this paper we extend this framework to a spacetime with a cosmological constant. By explicit construction we show that the Navier-Stokes equation can be derived from both black brane background and spatially curved spacetime. We also conjecture that imposing Petrov-like condition on the boundary should be equivalent to the conventional method using the hydrodynamical expansion of the metric in the near horizon limit.

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## I. INTRODUCTION

The hydrodynamical behavior of gravity has been greatly investigated since Damour discovered that the excitations of a black hole horizon behave very much like those of a fluid [1–10]. In particular, recent progress on the Wilsonian approach to Fluid/Gravity duality has opened a new window to link the Einstein equation to the Navier-Stokes equation for a general class of spacetime geometries [11–13]. In this setup the gravitational fluctuations are confined inside the region between the horizon and a finite cutoff at radius  $r = r_c$ , and a dual holographic fluid lives on the cutoff surface. The hydrodynamical quantities can be obtained in the non-relativistic limit through the standard procedure in AdS/CFT dictionary and their dependence on the cutoff  $r_c$  is interpreted as the renormalization group flow in the fluid [14–26].

The holographic nature of the Petrov-like boundary condition has been originally disclosed in [27]. It has been shown that embedding a hypersurface  $\Sigma_c$  into a Minkowski spacetime, a theory of gravity can be reduced to a theory of fluid without gravity in one less dimension by simply imposing Petrov-like condition on the boundary  $\Sigma_c$ . More explicitly, one finds that in near horizon limit such kind of boundary conditions will reduce the degrees of freedom of gravity such that the “momentum constraint” gives rise to the incompressible Navier-Stokes equation. In contrast to the conventional method using the hydrodynamical expansion of the metric, imposing such kind of boundary condition on the cutoff surface is much simpler and elegant than putting regularity condition on the horizon. In this approach the basic setup is to consider the perturbations of the extrinsic curvature of  $\Sigma_c$  while keeping the intrinsic metric fixed. Furthermore, in this approach we treat the Brown-York stress tensor as the fundamental variables, and identify its components with the hydrodynamical variables of a fluid directly. One even need not to know the explicit form of the perturbed metric in the bulk, thus no need to solve the perturbation equations in the bulk either. Recently, we have extended the framework above to a spacetime with a spatially curved embedding in [28].

In this paper we intend to further disclose the holographic nature of Petrov-like boundary condition by extending the framework to a spacetime with a cosmological constant. In this setting a black brane background is allowed. It is very known that in this kind of background the fluid/gravity duality has been extensively investigated through the traditional

hydrodynamic method with metric expansion in the bulk and various hydrodynamic parameters have been obtained for the dual fluid, see [16] and references therein. But it is completely unknown if imposing the Petrov-like condition on the cutoff surface would lead to the same results in this circumstance. We would like to stress that this is not a trivial issue at all because no one can say that these two methods are actually equivalent even in the near horizon limit. Technically, the former (namely the hydrodynamic expansion) imposes the regularity condition on the horizon and needs a long-wavelength expansion, while the latter (the Petrov-like condition) imposes the boundary condition on the cutoff surface and identifies the non-relativistic limit with the near horizon limit. Therefore, when a cosmological constant is taken into account, we are allowed to take the black brane as the background and then compare our results obtained by employing the Petrov-like conditions with the previous results obtained through the hydrodynamic expansion of the metric in the bulk. This is one of our main motivations for this paper. We will firstly present the Petrov-like boundary condition in terms of the Brown-York tensor and then consider the perturbations of the extrinsic curvature while keeping the intrinsic metric fixed. The Navier-Stokes equations with incompressibility conditions are derived from black brane background and spatially curved spacetime, respectively. In particular, for a black brane background we will discuss the consistency of our results with those previously obtained through the hydrodynamical expansion. The universality of imposing Petrov-like boundary condition in a general spacetime and its equivalence with the hydrodynamical expansion of the metric in the near horizon limit are proposed in the end of this paper.

## II. THE FRAMEWORK FOR A SPACETIME WITH A COSMOLOGICAL CONSTANT

For our purpose we assume the bulk metric in  $p + 2$  dimensional spacetime has a general form as

$$ds^2_{p+2} = -f(r)dt^2 + 2dtdr + e^{\rho(r, x^i)}\delta_{ij}dx^i dx^j, \quad (1)$$

where  $f$  is a function of radial coordinate  $r$ , while  $\rho$  depends on coordinates  $r$  and  $x^i$ . First of all, we require that this metric should be a solution to the vacuum Einstein equation with an arbitrary cosmological constant  $\Lambda$ , namely

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \quad \mu, \nu = 0, \dots, p + 1. \quad (2)$$

Plugging the above metric into Eq.(2) leads to the following equations

$$\begin{aligned}\partial_r \partial_i \rho &= 0 \\ \partial_r^2 \rho + \frac{1}{2}(\partial_r \rho)^2 &= 0 \\ \partial_r^2 f + \frac{p}{2}(\partial_r f)(\partial_r \rho) &= -\frac{4\Lambda}{p}\end{aligned}\tag{3}$$

and

$${}^{p+1}R_{ij} = \gamma_{ij} \left[ \frac{1}{2}(\partial_r f)(\partial_r \rho) + \frac{pf}{4}(\partial_r \rho)^2 + \frac{f}{2}\partial_r^2 \rho + \frac{2\Lambda}{p} \right].\tag{4}$$

It is straightforward to solve the equations in (3) and find the general solutions to be

$$\begin{aligned}f(r) &= -\frac{2\Lambda}{p(p+1)}(r+c)^2 - \frac{c_1}{(p-1)(r+c)^{p-1}} + c_2 \\ \rho(r, x^i) &= F(x^i) + 2\ln(r+c),\end{aligned}\tag{5}$$

where  $c$ ,  $c_1$  and  $c_2$  are integration constants. Here we need not to know the specific form of  $F(x^i)$ , which in principle can be obtained by solving equation (4).<sup>1</sup> In next two sections we will explicitly construct models with different backgrounds by choosing appropriate values for these constants.

Now we consider an embedding  $\Sigma_c$  with a  $p+1$  dimensional induced metric  $\gamma_{ab}$  by setting  $r = r_c$ . Since the spatial part of the hypersurface is conformally flat and

$${}^{p+1}R_{ij} = {}^pR_{ij} = \frac{c_2(p-1)}{(r_c+c)^2}\gamma_{ij},\tag{6}$$

one can show that the hypersurface must be a spacetime with a constant curvature with the use of the Einstein equation. On this cutoff surface its extrinsic curvature  $K_{ab}$  should satisfy the  $p+1$  dimensional ‘‘momentum constraint’’

$$D^a(K_{ab} - \gamma_{ab}K) = 0,\tag{7}$$

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<sup>1</sup> We would like to claim that non-trivial solutions for the function  $F(x^i)$  always exist. According to Eqs. (4), (5) and (6), we can obtain the constraint equation of  $F(x^i)$  as

$$\frac{c_2(p-1)}{(r_c+c)^2}\gamma_{ij} = \frac{2-p}{2}\partial_i\partial_j F - \frac{1}{2}\delta_{ij}\delta^{km}\partial_k\partial_m F + \frac{p-2}{4}(\partial_i F)(\partial_j F) - \frac{p-2}{4}\delta_{ij}\delta^{km}(\partial_k F)(\partial_m F).$$

In section III, the black brane metric with  $c_2 = 0$  and  $F(x^i) = 0$  satisfies the above equation automatically. Besides, we can always construct non-trivial solutions for spatially curved spacetime as discussed in section IV. For instance, the solutions could be  $F(x) = -2\ln x^i + \ln \frac{p(1-p)}{2\Lambda+p}$ , where  $x^i$  can be any spatial coordinate and  $\Lambda < -\frac{p}{2}$ . As a matter of fact, these solutions are nothing but the constant curvature hyperbolic space  $H^p$  with the Poincaré coordinates.

as well as the “Hamiltonian constraint”

$${}^{p+1}R + K_{ab}K^{ab} - K^2 - 2\Lambda = 0, \quad (8)$$

where  $D_a$  is compatible with the induced metric on  $\Sigma_c$ , namely  $D_a\gamma_{bc} = 0$ .

To impose Petrov-like boundary condition on this cutoff surface<sup>2</sup>, we decompose the  $p+2$  dimensional Weyl tensor in terms of  $p+1$  dimensional quantities such that the Petrov-like condition can be expressed in terms of the  $p+1$  dimensional curvature and the induced metric. Specifically, it can be decomposed as

$$\begin{aligned} C_{abcd} &= {}^{p+1}R_{abcd} + K_{ad}K_{bc} - K_{ac}K_{bd} + \frac{2\Lambda}{p(p+1)}(\gamma_{ad}\gamma_{bc} - \gamma_{ac}\gamma_{bd}) \\ C_{abc(n)} &= D_aK_{bc} - D_bK_{ac} \\ C_{a(n)c(n)} &= -{}^{p+1}R_{ac} + K K_{ac} - K_a{}^b K_{bc} + \frac{2\Lambda}{p+1}\gamma_{ac}, \end{aligned} \quad (9)$$

where  $\gamma_{ab} = g_{ab} - n_a n_b$ ,  $C_{abc(n)} = C_{abc\mu}n^\mu$ , and  $n^\mu$  is the unit normal to  $\Sigma_c$ . The Petrov-like boundary condition on  $\Sigma_c$  is defined as

$$C_{(\ell)i(\ell)j} = \ell^\mu m_i{}^\nu \ell^\alpha m_j{}^\beta C_{\mu\nu\alpha\beta} = 0, \quad (10)$$

where  $p+2$  Newman-Penrose-like vector fields satisfy the following relations

$$\ell^2 = k^2 = 0, \quad (k, \ell) = 1, \quad (k, m_i) = (\ell, m_i) = 0, \quad (m^i, m_j) = \delta^i_j. \quad (11)$$

The reduction from the Einstein equation to Navier-Stokes equation by imposing Petrov-like boundary condition can be understood by counting the degrees of freedom on the cutoff surface. Taking the extrinsic curvature as the fundamental variables, we have  $(p+1)(p+2)/2$  independent components, while the Petrov-like boundary condition puts  $p(p+1)/2 - 1$  constraints on the extrinsic curvature, where we need subtract one simply as the Weyl tensor is traceless. Therefore, the remaining degrees of freedom is  $p+2$ , which may be identified as the energy density, pressure and the velocity of the dual fluid living on the

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<sup>2</sup> Traditionally the Petrov conditions are used to classify the geometry of spacetime by considering the multiplicity of principle null vector at each point of spacetime. In fact, the Petrov condition is a point-wise definition, so it can be generalized to a hypersurface, i.e. only consider points on that hypersurface. In this paper, Eq.(10) is just the condition we impose on the boundary, which is exactly the same as that in [27]. Nevertheless, to avoid possible confusion we would like to rename it as Petrov-like boundary condition through this paper.

cutoff surface. In this sense the relations between the pressure and the velocity of the fluid is governed by the  $p + 1$  momentum constraints (7), while the Hamiltonian constraint (8) is an equation relating the energy density to the pressure of the fluid. In next two sections we will explicitly demonstrate that  $p + 1$  momentum constraints on the cutoff surface will give rise to the incompressibility condition and the Navier-Stokes equation in the near horizon limit.

### III. NAVIER-STOKES EQUATION FROM A BLACK BRANE BACKGROUND

In this section we will employ the Petrov-like boundary condition to a black brane background and then derive the Navier-Stokes equation on the cutoff surface in the near horizon limit. Previously this result has been obtained with a hydrodynamical expansion of the metric in [16]. Setting the integration constants in Eq.(5) as

$$\begin{aligned} c_1 &= 2m(p-1), \quad c_2 = 0, \quad c = 0, \\ \Lambda &= -\frac{p(p+1)}{2}, \quad F(x^i) = 0, \end{aligned} \tag{12}$$

we can obtain a black brane metric as

$$ds^2_{p+2} = -f(r)dt^2 + 2dtdr + r^2\delta_{ij}dx^i dx^j, \tag{13}$$

where  $f(r) = r^2(1 - \frac{r_h^{p+1}}{r^{p+1}})$  and  $r_h$  is the position of the horizon. The hypersurface  $\Sigma_c$  is located outside the horizon at  $r = r_c$ , then the induced metric on the hypersurface is

$$\begin{aligned} ds^2_{p+1} &= -f(r_c)dt^2 + r_c^2\delta_{ij}dx^i dx^j \\ &\equiv -dx^{02} + r_c^2\delta_{ij}dx^i dx^j, \end{aligned} \tag{14}$$

where we have defined  $x^0 = \sqrt{f(r_c)}t$ . Obviously this is a spatially flat embedding. In order to investigate the hydrodynamical behavior of geometry in the non-relativistic limit, we further introduce a parameter  $\lambda$  by rescaling the time coordinate with  $\tau = \lambda x^0$  such that the metric has the form

$$ds^2_{p+1} = -\frac{1}{\lambda^2}d\tau^2 + r_c^2\delta_{ij}dx^i dx^j. \tag{15}$$

The non-relativistic limit is achieved by setting  $\lambda \rightarrow 0$ . In this coordinate system, the components of the extrinsic curvature are

$$\begin{aligned} K^\tau{}_\tau &= \frac{1}{2\sqrt{f}}\partial_r f, & K^\tau{}_i &= 0, \\ K^i{}_j &= \frac{1}{r}\sqrt{f}\delta^i{}_j, & K &= \frac{1}{2\sqrt{f}}\partial_r f + \frac{p}{r}\sqrt{f}, \end{aligned} \quad (16)$$

where  $K$  is the trace of the extrinsic curvature. Employing the Brown-York stress tensor which is defined on  $\Sigma_c$  as

$$t_{ab} = K\gamma_{ab} - K_{ab}, \quad (17)$$

we can further express the extrinsic curvature in terms of the Brown-York tensor as follows

$$\begin{aligned} K^\tau{}_\tau &= \frac{t_{tr}}{p} - t^\tau{}_\tau, & K^\tau{}_i &= -t^\tau{}_i, \\ K^i{}_j &= -t^i{}_j + \delta^i{}_j \frac{t_{tr}}{p}, & K &= \frac{t_{tr}}{p}, \end{aligned} \quad (18)$$

where  $t_{tr}$  is the trace of  $t_{ab}$ . Now with the setup presented above, we start to investigate the dynamical behavior of the gravity on  $\Sigma_c$  in the near horizon limit. First of all, in contrast to the conventional perturbation method with metric expansion, here we take the extrinsic curvature of the cutoff surface as the fundamental variable while keeping the intrinsic metric of the surface fixed. More conveniently, we may directly consider the fluctuations of the Brown-York tensor on the surface, and expand its components in powers of  $\lambda$  as

$$\begin{aligned} t^\tau{}_i &= 0 + \lambda t^\tau{}_i^{(1)} + \dots \\ t^\tau{}_\tau &= \frac{p}{r}\sqrt{f} + \lambda t^\tau{}_\tau^{(1)} + \dots \\ t^i{}_j &= \left(\frac{1}{2\sqrt{f}}\partial_r f + \frac{p-1}{r}\sqrt{f}\right)\delta^i{}_j + \lambda t^i{}_j^{(1)} + \dots \\ t_{tr} &= \left(\frac{p}{2\sqrt{f}}\partial_r f + \frac{p^2}{r}\sqrt{f}\right) + \lambda t_{tr}^{(1)} + \dots \end{aligned} \quad (19)$$

Now we concentrate on the perturbation behavior of gravity in the near horizon limit. Mathematically the equivalence between the long wavelength hydrodynamical expansion and the near horizon expansion has been stressed in [14], even at the nonlinear level. Here we find the near horizon limit can be achieved simultaneously with the non-relativistic limit through relating the perturbation parameter  $\lambda$  to  $(r_c - r_h)$ , namely, the coordinate distance of the cutoff surface to the horizon. More explicitly, we find they may be related by

$\lambda^2 = \alpha(r_c - r_h)$ , with  $\alpha = \frac{1}{(p+1)r_h}$ , which also implies that  $\tau = (r_c - r_h)t$ .<sup>3</sup> As a consequence, the near horizon expansion of the functions in Brown-York tensor can be expressed in powers of  $\lambda$  as follows

$$\begin{aligned}\frac{\partial_r f}{\sqrt{f}}|_{r_c} &= \frac{1}{\lambda} - \frac{3(p-2)(p+1)}{4}\lambda + \dots \\ \frac{\sqrt{f}}{r}|_{r_c} &= (1+p)\lambda - \frac{(p+2)(p+1)^2}{4}\lambda^3 + \dots \\ \sqrt{f}|_{r_c} &= r_h(p+1)\lambda + \frac{r_h(p+1)}{4}(2+p-p^2)\lambda^3 + \dots\end{aligned}\quad (20)$$

Mathematically this identification leads to the mixing of the near horizon expansion with the non-relativistic expansion in the fluctuations. It is interesting enough to notice that this kind of mixing plays an essential role in deriving the standard Navier-Stokes equation with unit shear viscosity in this coordinate system, and we will show this immediately. Firstly, we rewrite the Hamiltonian constraint in terms of the Brown-York tensor as

$$(t^\tau_\tau)^2 - \frac{2}{\lambda^2}\gamma^{ij}t^\tau_i t^\tau_j + t^i_j t^j_i - \frac{(t_{tr})^2}{p} - 2\Lambda = 0. \quad (21)$$

Taking the expansion in powers of  $\lambda$ , we find the non-trivial leading order of this equation gives rise to

$$t^\tau_\tau^{(1)} = -2\gamma^{ij}t^\tau_i^{(1)}t^\tau_j^{(1)}. \quad (22)$$

By choosing the vector fields as

$$\sqrt{2}l = \partial_0 - n, \quad \sqrt{2}k = -\partial_0 - n, \quad m_i = \frac{1}{r}\partial_i, \quad (23)$$

similarly, with the use of Eq.(9) the Petrov-like condition can be rewritten in terms of the Brown-York tensor as follows

$$t^\tau_\tau t^i_j + \frac{2}{\lambda^2}\gamma^{ik}t^\tau_k t^\tau_j - 2\lambda t^i_{j,\tau} - t^i_k t^k_j - \frac{2}{\lambda}\gamma^{ik}t^\tau_{(k,j)} + \delta^i_j \left[ \frac{t_{tr}}{p} \left( \frac{t_{tr}}{p} - t^\tau_\tau \right) + \frac{2\Lambda}{p} + 2\lambda \partial_\tau \frac{t_{tr}}{p} \right] = 0. \quad (24)$$

Expanding the equation in powers of  $\lambda$ , we have

$$t^i_j^{(1)} = 2\gamma^{ik}t^\tau_k^{(1)}t^\tau_j^{(1)} - 2\gamma^{ik}t^\tau_{(k,j)}^{(1)} + \delta^i_j \frac{t_{tr}^{(1)}}{p}. \quad (25)$$

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<sup>3</sup> The parameter  $\alpha$  can be fixed as follows. We firstly change the position of the horizon to  $r' = 0$  by a translation  $r' = r - r_h$ , and then expand the bulk metric of the black brane near the horizon, leading to  $ds^2 = -(p+1)r_h r' dt^2 + 2dt dr' + r_h^2 dx_i dx^i$ . Further taking a coordinate transformation by  $\tilde{t} = (p+1)r_h t$  and  $\tilde{r} = \frac{r'}{(p+1)r_h}$ , we find it is nothing but the Rindler wedge of a Minkowski space. Since for the Rindler wedge, we identify  $\lambda^2$  with the location of the hyperbola by  $\lambda^2 = \tilde{r}_c$ ,  $\tau = \lambda^2 \tilde{t}$ , for the black brane it corresponds to setting  $\lambda^2 = \frac{r_c - r_h}{(p+1)r_h}$  in  $(t, r)$  coordinate system.



The momentum constraint in flat cutoff surface has the form

$$\partial_a t^a_b = 0. \quad (26)$$

Plugging the solutions obtained in Eq.(25) into this equation and identifying

$$t^\tau_i{}^{(1)} = \frac{1}{2}v_i, \quad t_{tr}{}^{(1)} = \frac{p}{2}P, \quad (27)$$

we can straightforwardly obtain the incompressibility condition and the standard Navier-Stokes equation with unit shear viscosity as

$$\partial_i v^i = 0, \quad (28)$$

$$\partial_\tau v_i + v^k \partial_k v_i - \partial^2 v_i + \partial_i P = 0. \quad (29)$$

In the end of this section we remark that our results are consistent with the previous results presented in Ref.[16], although the fluid quantities are identified with gravity quantities in a different manner and the Navier-Stokes equation looks not exactly identical. As a matter of fact, the apparent discrepancy comes from the use of different coordinate systems. Identifying the coordinates  $\tau$ ,  $\tilde{\tau}$  and  $x^i$  in [16] with ours  $t$ ,  $x^0$  and  $x^i$  respectively, the incompressible Navier-Stokes equation obtained for arbitrary finite cutoff  $r = r_c$  in [16] becomes

$$\partial_0 \beta_i - \nu_c \partial^2 \beta_i + \partial_i P_c + \beta^j \partial_j \beta_i = 0, \quad (30)$$

where the viscosity  $\nu_c = \frac{r_c}{r_h(1+p)}(1 - \frac{r_h^{p+1}}{r_c^{p+1}})^{1/2}$ . Apparently  $\nu_c$  is vanishing in the near horizon limit. However, if we transform the coordinate system to  $(\tau, x^i)$  by rescaling

$$x^0 = \frac{1}{\lambda}\tau, \quad (31)$$

the equation then becomes

$$\lambda \partial_\tau \beta_i - \nu_c \partial^2 \beta_i + \partial_i P_c + \beta^j \partial_j \beta_i = 0. \quad (32)$$

Furthermore, when the time is rescaled as Eq.(31), by definition the velocity and the pressure of the fluid should correspondingly be rescaled as

$$\beta_i = \lambda v_i, \quad P_c = \lambda^2 P. \quad (33)$$

Thus, Eq.(32) changes into the form

$$\partial_\tau v_i - \frac{\nu_c}{\lambda} \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0, \quad (34)$$

where  $v_i$  and  $P$  are exactly the velocity and pressure of the fluid as shown before. It is easy to see that in the limit  $r_c \rightarrow r_h$ ,  $\frac{\nu_c}{\lambda} = 1$  and one obtains the standard incompressible Navier-Stokes equation with unit shear viscosity, which is the same as what we have obtained. Finally, it is worthy to point out that in [16] this equation is obtained for arbitrary finite cutoff but here we have only considered the near horizon limit, thus one may also notice that in the hydrodynamical expansion of the metric in the bulk, the process depends on the number of spatial dimension  $p$  and the Navier-Stokes equation is derived only for some specific  $p$ , for instance  $p = 3$  [16]. However, in our case since the near horizon limit is taken, we find the result is general and independent of the spatial dimension  $p$ .

#### IV. NAVIER-STOKES EQUATION IN SPATIALLY CURVED SPACETIME

In this section we intend to derive the Navier-Stokes equations by imposing Petrov-like boundary conditions in a spacetime with non-vanishing Ricci tensor  ${}^p R_{ij}$ . First of all, for convenience we prefer to fix the Rindler horizon at  $r = 0$ . This can be easily implemented by choosing the integration constants in Eq.(5) as follows

$$c = 1, \quad c_1 = 1 + \frac{4\Lambda}{p(p+1)}, \quad c_2 = \frac{2\Lambda}{(p-1)p} + \frac{1}{p-1}. \quad (35)$$

With this choice we find the functions  $f(r)$  and  $\rho(r, x^i)$  can be expanded in the near horizon limit as

$$\begin{aligned} f(r) &= r - \left(\frac{p}{2} + \frac{2\Lambda}{p}\right)r^2 + \frac{p^2 + p + 4\Lambda}{6}r^3 + \dots \\ \rho(r, x^i) &= F(x^i) + 2r - r^2 + \frac{2}{3}r^3 + \dots \end{aligned} \quad (36)$$

Now we consider an embedded hypersurface located at  $r = r_c$ . The induced metric on this surface is

$$\begin{aligned} ds_{p+1}^2 &= -f(r_c)dt^2 + e^\rho \delta_{ij} dx^i dx^j \\ &\equiv -dx^{02} + e^\rho \delta_{ij} dx^i dx^j \\ &\equiv -\frac{d\tau^2}{\lambda^2} + e^\rho \delta_{ij} dx^i dx^j, \end{aligned} \quad (37)$$

where a parameter  $\lambda$  is introduced to discuss the dynamical behavior of gravity on  $\Sigma_c$  in the non-relativistic limit. Moreover, we identify the parameter  $\lambda$  with the location of

hypersurface by setting  $r_c = \lambda^2$  such that a large mean curvature can be satisfied in the near-horizon limit  $\lambda \rightarrow 0$ . In coordinate system  $(\tau, x^i)$ , the non-zero components of the connection are

$$\Gamma^k_{ij} = \frac{1}{2}(\delta^k_j \partial_i \rho + \delta^k_i \partial_j \rho - \delta^{km} \delta_{ij} \partial_m \rho), \quad (38)$$

which depend on the specific form of  $F(x^i)$  and in general describe a spatially curved space-time. Now we turn to consider the perturbations of gravity on this background. The extrinsic curvature of the cutoff surface is

$$\begin{aligned} K^\tau_i &= 0, & K^\tau_\tau &= \frac{\partial_r f}{2\sqrt{f}}, \\ K^i_j &= \frac{1}{2}\sqrt{f}\partial_r \rho \delta^i_j, & K &= \frac{1}{2\sqrt{f}}\partial_r f + \frac{1}{2}p\sqrt{f}\partial_r \rho. \end{aligned} \quad (39)$$

Similarly we consider the fluctuations of the Brown-York tensor and expand it in powers of  $\lambda$  as

$$\begin{aligned} t^\tau_i &= 0 + \lambda t^\tau_i{}^{(1)} + \dots \\ t^\tau_\tau &= \frac{p}{2}\sqrt{f}\partial_r \rho + \lambda t^\tau_\tau{}^{(1)} + \dots \\ t^i_j &= \left(\frac{\partial_r f}{2\sqrt{f}} + \frac{p-1}{2}\sqrt{f}\partial_r \rho\right)\delta^i_j + \lambda t^i_j{}^{(1)} + \dots \\ t_{tr} &= \left(\frac{p}{2\sqrt{f}}\partial_r f + \frac{p^2}{2}\sqrt{f}\partial_r \rho\right) + \lambda t_{tr}{}^{(1)} + \dots \end{aligned} \quad (40)$$

Moreover, since the location of the hypersurface  $r_c$  is identified with  $\lambda^2$ , we expand the following quantities in powers of  $r_c$

$$\begin{aligned} \partial_r \rho|_{r_c} &= 2 - 2r_c + 2r_c^2 + \dots \\ \partial_r f|_{r_c} &= 1 - \left(p + \frac{4\Lambda}{p}\right)r_c + \dots \\ f|_{r_c} &= r_c - \left(\frac{p}{2} + \frac{2\Lambda}{p}\right)r_c^2 + \dots \\ \sqrt{f}|_{r_c} &= r_c^{1/2} - \left(\frac{p}{4} + \frac{\Lambda}{p}\right)r_c^{3/2} + \dots \\ \frac{\partial_r f}{\sqrt{f}}|_{r_c} &= r_c^{-1/2} - \frac{3(p^2 + 4\Lambda)}{4p}r_c^{1/2} + \dots \end{aligned} \quad (41)$$

Next we turn to expand the Hamiltonian constraint as well as the Petrov-like boundary conditions in powers of  $\lambda$ . Substituting Eq.(18) into Eq.(8), we find the Hamiltonian constraint becomes

$$^{p+1}R + (t^\tau_\tau)^2 - \frac{2}{\lambda^2}\gamma^{ij}t^\tau_i t^\tau_j + t^i_j t^j_i - \frac{(t_{tr})^2}{p} - 2\Lambda = 0. \quad (42)$$

It is easy to check that the background satisfies this condition automatically at the order of  $\lambda^{-2}$ . While the non-trivial sub-leading order is  $\lambda^0$  which gives rise to

$$t^\tau_{\tau}{}^{(1)} = -2\gamma^{ij(0)}t^\tau_i{}^{(1)}t^\tau_j{}^{(1)}. \quad (43)$$

Here we have used  ${}^{p+1}R^{(0)} = 2\Lambda + p$  and expanded the spatial metric  $\gamma_{ij}$  as

$$\gamma_{ij} = e^{F(x^i)}\delta_{ij}(1+r)^2 \equiv \gamma_{ij}^{(0)} + r\gamma_{ij}^{(1)} + r^2\gamma_{ij}^{(2)}. \quad (44)$$

Thus  $\gamma_{ij}^{(0)} \equiv e^{F(x^i)}\delta_{ij}$ . As we have pointed out in [28], the ‘‘spatially covariant derivative’’  $D_i$  compatible with  $\gamma_{ij}$  is also compatible with  $\gamma_{ij}^{(n)}$  since the connection is  $r$ -independent. We will use this fact when deriving the Navier-Stokes equation from the momentum constraint. Now we turn to the Petrov-like boundary condition. We choose the vector fields as

$$\sqrt{2}l = \partial_0 - n, \quad \sqrt{2}k = -\partial_0 - n, \quad m_i = e^{\frac{-p}{2}}\partial_i, \quad (45)$$

then with the use of Eq.(9), we find the boundary condition Eq.(10) can be finally written in terms of the Brown-York stress tensor as well as the intrinsic curvature as

$$\begin{aligned} t^\tau_{\tau}t^k_j + \delta^k_j\left[\frac{t_{tr}}{p}\left(\frac{t_{tr}}{p} - t^\tau_{\tau}\right) + \frac{2\lambda}{p}\partial_\tau t_{tr}\right] + \frac{2}{\lambda^2}\gamma^{ki}t^\tau_i t^\tau_j - 2\lambda t^k_{j,\tau} \\ - \frac{2\gamma^{ki}}{\lambda}t^\tau_{(i,j)} - t^k_m t^m_j - \gamma^{ki}R_{ij} + \frac{2\gamma^{ki}}{\lambda}\Gamma^m_{ij}t^\tau_m + \frac{2\Lambda}{p}\delta^k_j = 0. \end{aligned} \quad (46)$$

Similarly it is easy to check that the background satisfies this condition automatically at the order of  $\lambda^{-2}$ , while at the order of  $\lambda^0$  we obtain the following result

$$t^k_j{}^{(1)} = 2\gamma^{ki(0)}t^\tau_i{}^{(1)}t^\tau_j{}^{(1)} - 2\gamma^{ki(0)}t^\tau_{(i,j)}{}^{(1)} + \frac{t_{tr}{}^{(1)}}{p}\delta^k_j + 2\gamma^{ki(0)}\Gamma^m_{ij}t^\tau_m{}^{(1)}. \quad (47)$$

Next we put this solution into the momentum constraint to reduce the degrees of freedom of gravity. Now the momentum constraint on  $\Sigma_c$  is

$$D_a t^a{}_b = 0. \quad (48)$$

The time component of this equation at leading order gives rise to

$$D_i t^{\tau i(1)} = 0, \quad (49)$$

while the spatial components can be written as

$$\partial_\tau t^{\tau i(1)} + D_k t^{ki(1)} = 0. \quad (50)$$

Plugging the solution in Eq.(47) into this equation, we have

$$\partial_\tau t_i^{\tau(1)} + 2t^{\tau k(1)} D_k t_i^{\tau(1)} + \frac{1}{p} D_i t_{tr}^{(1)} - (D^k D_k t_i^{\tau(1)} + t_m^{\tau(1)} R^m_i) = 0. \quad (51)$$

Similarly, we identify the remaining Brown-York variables with the hydrodynamical variables as

$$t_{tr}^{(1)} = \frac{p}{2} P, \quad t_i^{\tau(1)} = \frac{v_i}{2}, \quad (52)$$

then the incompressibility condition and Navier-Stokes equation in spatially curved space-time can be obtained as follows

$$D_i v^i = 0, \quad (53)$$

$$\partial_\tau v_i + v^k D_k v_i + D_i P - (D^k D_k v_i + R^m_i v_m) = 0. \quad (54)$$

## V. SUMMARY AND DISCUSSIONS

In this paper we have generalized the framework presented in [27, 28] to a spacetime with a cosmological constant. We have demonstrated that the incompressible Navier-Stokes equation can be derived from the Einstein equation by simply imposing the Petrov-like boundary condition in the near horizon limit such that the holographic nature of the Petrov-like boundary condition has been further disclosed. Furthermore, we have shown that our results are consistent with the ones previously obtained by hydrodynamic expansions for a black brane background. We remark that the fundamental variables such as the velocity and the pressure for a fluid are introduced in different manners for these two methods. For hydrodynamic expansion the velocity of the fluid is identified with the fluctuations of the metric, while for Petrov-like boundary condition it is identified with the component of the Brown-York tensor. One may wonder why these different identifications do give rise to the same dynamical equations for a fluid. For a black brane background we have presented a detailed comparison between the method of Petrov-like boundary condition and the method of the hydrodynamic expansion of the metric in the near horizon limit. After having figured out the connections of those hydrodynamic quantities which are identified with gravity quantities in different manners in these two methods, we find the values of the shear viscosity obtained through these two methods are exactly identical such that the reliability of the gravity/fluid duality has been testified in a more solid foundation. On

the other hand, this consistency indicates that Petrov-like boundary condition contains a holographic nature indeed, with some universality in linking the Einstein equation to the Navier-Stokes equation. Our observation here is crucial for us to better understand these two methods and finally to be able to prove the conjecture that they may be equivalent in the near horizon limit. The next step is to testify its validity in a more general setting. First of all, we have obtained the Navier-Stokes equation on a spatially curved hypersurface, but only for a class of spacetime with constant curvature, we expect this framework can be applied to a background with a more general metric than what we have proposed in Eq.(1). Secondly, when a matter field is taken into account, besides extending the framework to including the contribution from the matter field in the Petrov-like boundary condition, it also involves how to put appropriate boundary conditions on the cutoff surface for the matter field itself when it is also dynamical and has some degrees of freedom. We will construct a model with Maxwell field and provide an affirmative answer to this issue elsewhere [29]. Therefore, based on all investigations mentioned above we may conjecture that at least in the near horizon limit, imposing Petrov-like condition on the boundary should be a universal method to reduce the Einstein equation to the Navier-Stokes equation for a general spacetime in the presence of a horizon. Furthermore, in such a limit it may be equivalent to the conventional method using the hydrodynamical expansion of the metric, where the perturbation equations in the bulk are explicitly studied and the regularity condition is imposed on the horizon. We leave this open issue for further investigation in future.

Since the higher order hydrodynamical expansion has been considered in [15], and the higher order corrections to the Navier-Stokes equations have also been derived there, we think it is an interesting issue to study the higher order expansions of the boundary conditions and constraints in the approach of imposing Petrov-like boundary conditions. Such investigations would provide us more understandings on correction terms in the Navier-Stokes equation for real fluids.

In this approach the mixing of the perturbation expansion and the near horizon expansion play a very interesting role in obtaining the desired results. On the other hand, as we know the Navier-Stokes equation can be derived on arbitrary finite cutoff surface with the method of long wavelength hydrodynamical expansion, and this approach is mathematically equivalent to the near horizon expansion even at the nonlinear level [14]. Thus we are wondering if the method of imposing Petrov-like boundary condition can be applicable to

the regime beyond the near horizon limit, and our investigation is under progress.

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